A Lower Bound on the Rational Degree of Boolean Functions

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Introduction

A surprisingly powerful tool for analyzing and proving the limits of quantum algorithms is the polynomial method. The core insight, developed by Beals et al. (1998), is that the final acceptance probability of any T-query quantum algorithm can be described perfectly by a real polynomial of the input variables with a degree of at most $2T$.

This connection gives us the famous lower bound $Q(f) \geq \deg(f)/2$. By finding the degree of a function's representative polynomial, a purely mathematical task, we can establish a hard limit on how fast any quantum algorithm can possibly be. This method has been instrumental in proving foundational results, such as the optimality of Grover's search algorithm.

Problem Setup

At its core, computer science studies Boolean functions, operations that take a series of binary inputs and produce a single binary output. To analyze their complexity, we can represent them using mathematical objects like polynomials.

- Fourier degree ($\deg(f)$): Every Boolean function can be uniquely written as a multilinear polynomial with real coefficients. For example, the AND function on two variables ($x_1 \wedge x_2$) can be represented by the polynomial $x_1 x_2$. The **Fourier degree** is the degree of such polynomial.
- Rational degree ($\mathrm{rdeg}(f)$): Instead of using a polynomial, we can also represent a function using a rational function $P(x)/Q(x)$. The **rational degree** is the minimum possible degree of such a representation.

The Rational Degree Conjecture

In 1994, Nisan and Szegedy posed the Rational Degree Conjecture, which states that the two measures are "polynomially related." This means that the rational degree can't be drastically smaller than the Fourier degree: $\deg(f) \leq (\mathrm{rdeg}(f))^c$ for some constant c.

Based on patterns in specific Boolean functions (Symmetric, Monotone, Unate, etc.), a more concrete conjecture has emerged:

$$\Omega(\sqrt{\deg(f)}) \leq \mathrm{rdeg}(f).$$

Connection to Quantum Computation

Later work revealed an interesting connection: the rational degree exactly captures the quantum query complexity with the power of post-selection. Post-selection is a theoretical model where we can discard all the failed runs of an algorithm and only consider the successful ones. Answering the conjecture would tell us how much computational power this ability gives a quantum computer compared to standard models.

Main Result: A New Lower Bound

While the full conjecture remains open, our work establishes the first non-trivial lower bound.

We proved that for any Boolean function f:

$$\Omega(\sqrt{\log \deg(f)}) \leq \mathrm{rdeg}(f).$$

This result is a consequence of a more general bound we proved for any Boolean function that depends on n input variables:

$$\Omega(\sqrt{\log n}) \leq \mathrm{rdeg}(f).$$

Remarks

Other major conjectures in the field, such as the Gotsman-Linial Conjecture, suggest that a stronger relationship holds. If true, these conjectures would imply a tighter lower bound of:

$$\Omega(\log n) \leq \mathrm{rdeg}(f).$$

This bound is believed to be optimal, as the address function achieves this lower bound.

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