Minimal Surfaces and the Weierstrass-Enneper Representation

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Outline

- Surfaces and Minimal Surfaces
- Weierstrass-Enneper Representation

Surfaces and Minimal Surfaces

- Background on Surfaces in $\mathbb{R}^n$
- Isothermal parametrization
- Minimal Surfaces

Background on Surfaces in $\mathbb{R}^n$

Definition

Let $U \subset \mathbb{R}^2$ be an open set, a regular parametrized surfaces is a map:

$$f : U \to \mathbb{R}^n$$

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Definition

Let Σ be a regular parametrized surface $f : U \to \mathbb{R}^n$, define the area of Σ:

$$A(\Sigma) = \iint_\Sigma dA = \iint_U \left| \frac{\partial f}{\partial u_1} \wedge \frac{\partial f}{\partial u_2} \right| dudv$$

Background on Surfaces in $\mathbb{R}^n$

Principal curvatures k_1, k_2, Gaussian curvature $K = k_1 k_2$, and mean curvature $H = \frac{k_1+k_2}{2}$.

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Let $b_{ij}(N) = \langle \frac{\partial^2 f}{\partial u_i \partial u_j}, N \rangle$, $g_{ij} = \langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \rangle$, $i, j = 1, 2$,
then

$$H = \frac{1}{2} \frac{b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11}}{g_{11}g_{22} - g_{12}^2}$$

Isothermal parametrization

Definition

Let $U \subset \mathbb{R}^2$, $f : U \to \mathbb{R}^n$ be a regular parametrized surface.
$f(u_1, u_2) = (x_1(u_1, u_2), \ldots, x_n(u_1, u_2))$.
If the parameters u_1, u_2 satisfy:

$$\left|\frac{\partial f}{\partial u_1}\right| = \left|\frac{\partial f}{\partial u_2}\right| > 0, \left\langle \frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2} \right\rangle = 0$$

Then they are called isothermal parameters.

Let $\lambda(u) = \left|\frac{\partial f}{\partial u_1}\right| = \left|\frac{\partial f}{\partial u_2}\right|$.

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$$H = \frac{1}{2} \frac{b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11}}{g_{11}g_{22} - g_{12}^2} = \frac{b_{11}(N) + b_{22}(N)}{2\lambda^2}$$

Minimal Surfaces

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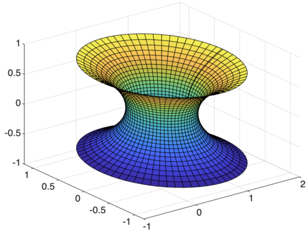
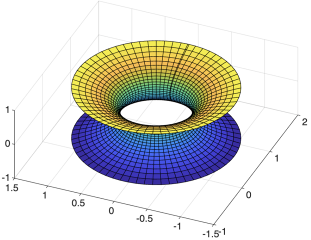
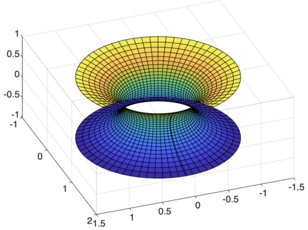
$$A'(0) = -\iint_{\Sigma} 2hHdA$$

Minimal Surfaces Examples

$$f(u,v) = (\cosh(u)\cos(v), \cosh(u)\sin(v), u)$$

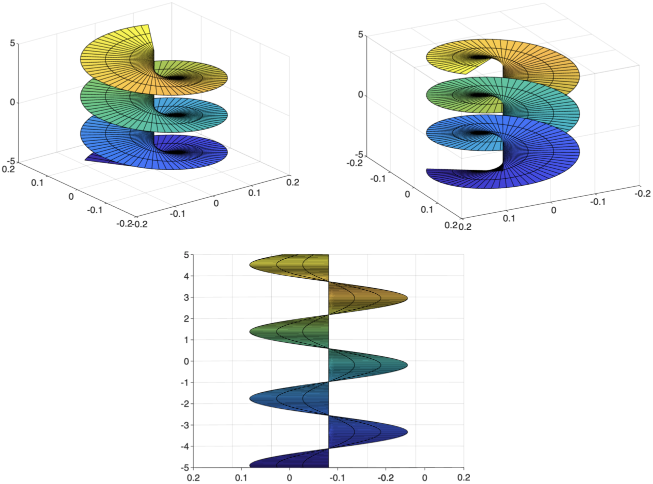






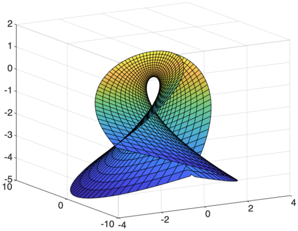
Minimal Surfaces Examples

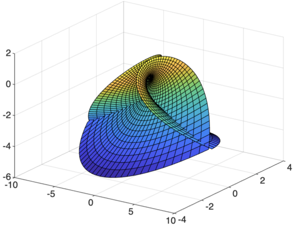
$$f(u,v) = (\sinh(u)\sin(v), -\sinh(u)\cos(v), -v)$$

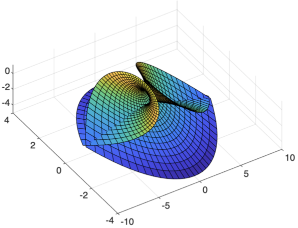


Minimal Surfaces Examples

$$f(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, -v + \frac{1}{3}v^3 - vu^2, u^2 - v^2\right)$$

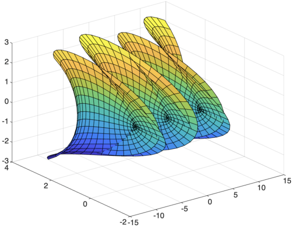


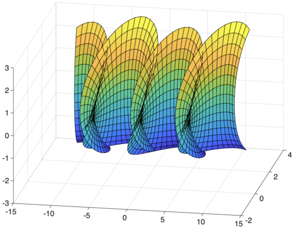


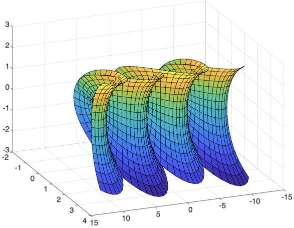


Minimal Surfaces Examples

$$f(u,v) = (u - \sin(u)\cosh(v), 1 - \cos(u)\cosh(v), 4\sin(\frac{u}{2})\sinh(\frac{v}{2}))$$







Weierstrass-Enneper Representation

Definition (Weierstrass-Enneper Representation)

Let D be a domain in the complex plane, $g(z)$ an arbitrary meromorphic function in D and $f(z)$ an analytic function in D having the property that fg^2 is an analytic function in D. Let

$$\varphi_1 = f(1 - g^2),\ \varphi_2 = if(1 + g^2),\ \varphi_3 = 2fg$$

Every nonplanar minimal surface defined over a simply connected domain can be represented in the form:

$$x_j(\zeta) = \mathrm{Re}\left\{\int_0^\zeta \varphi_j(z)dz\right\} + c_j,\ j = 1, 2, 3$$

Weierstrass-Enneper Representation

Step 1:
Let $f : \mathbb{R}^2 \to \mathbb{R}$, $z = u + iv, \bar{z} = u - iv$, we have $u = \frac{z+\bar{z}}{2}, v = \frac{z-\bar{z}}{2i}$. By chain rule:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial u} - i\frac{\partial f}{\partial v}\right), \ \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial u} + i\frac{\partial f}{\partial v}\right)$$

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If $f(u,v) = a(u,v) + ib(u,v)$, then the Cauchy-Riemann equations are satisfied if and only if:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\left(\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v}\right) + i\left(\frac{\partial a}{\partial v} + \frac{\partial b}{\partial u}\right)\right) = 0$$

Weierstrass-Enneper Representation

Step 2:
Let Σ be a minimal surface with isothermal parametrization $x(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v))$.
Let $z = u + iv$, we obtain $x(z, \bar{z}) = (x^1(z, \bar{z}), x^2(z, \bar{z}), x^3(z, \bar{z}))$, and let

$$\varphi = \frac{\partial x}{\partial z}, \ \varphi_j = \frac{\partial x^j}{\partial z}$$

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$$\varphi = \frac{\partial x}{\partial z}, \ \varphi_j = \frac{\partial x^j}{\partial z}$$

$$\frac{\partial \varphi_j}{\partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2 x^j}{\partial u^2} + \frac{\partial^2 x^j}{\partial v^2}\right)$$

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$$\begin{aligned}
\langle \varphi, \varphi \rangle = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 &= \frac{1}{4}\sum_{j=1}^{3}\left(\frac{\partial x^j}{\partial u} - i\frac{\partial x^j}{\partial v}\right)^2 \\
&= \frac{1}{4}\sum_{j=1}^{3}\left[\left(\frac{\partial x^j}{\partial u}\right)^2 - \left(\frac{\partial x^j}{\partial v}\right)^2 - 2i\frac{\partial x^j}{\partial u}\frac{\partial x^j}{\partial v}\right] = 0
\end{aligned}$$

Weierstrass-Enneper Representation

Step 3:

$$\begin{aligned}
dx^j &= \frac{\partial x^j}{\partial z}dz + \frac{\partial x^j}{\partial \bar{z}}d\bar{z} \\
&= \varphi_j dz + \bar{\varphi}_j d\bar{z} \\
&= \frac{1}{2}(x_u^j - ix_v^j)(du + idv) + \frac{1}{2}(x_u^j + ix_v^j)(du - idv) \\
&= x_u^j du + x_v^j dv \\
&= 2\,\mathrm{Re}(\varphi_j dz)
\end{aligned}$$

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&= x_u^j du + x_v^j dv \\
&= 2\,\mathrm{Re}(\varphi_j dz)
\end{aligned}
$$

$$
x^j(u, v) = 2\,\mathrm{Re}\left\{ \int_\Gamma \varphi_j(z) dz \right\} + c_j
$$

Weierstrass-Enneper Representation

Step 4:
Let $f = \varphi_1 - i\varphi_2$, $g = \varphi_3/(\varphi_1 - i\varphi_2)$, we have:

$$\begin{aligned}
f(1-g^2) &= (\varphi_1 - i\varphi_2)\left(1 - \frac{\varphi_3}{\varphi_1 - i\varphi_2}\right) \\
&= \varphi_1 - i\varphi_2 + \varphi_1 + i\varphi_2 \\
&= 2\varphi_1 \\
if(1+g^2) &= i(\varphi_1 - i\varphi_2)\left(1 + \frac{\varphi_3}{\varphi_1 - i\varphi_2}\right) \\
&= i\varphi_1 + \varphi_2 - i(\varphi_1 + i\varphi_2) \\
&= 2\varphi_2 \\
2fg &= 2\varphi_3
\end{aligned}$$

Weierstrass-Enneper Representation Examples

- Helicoid:
$$f(z) = \frac{i}{2e^z}, \; g(z) = e^z$$
- Catenoid:
$$f(z) = \frac{1}{2e^z}, \; g(z) = e^z$$
- Enneper:
$$f(z) = 1, \; g(z) = z$$
- Catalan:
$$f(z) = 1 - e^{iz}, \; g(z) = \frac{2\sinh(-\frac{iz}{2})}{1 - e^{iz}}$$

My Research Project

Regularity theory of boundary of plateau problem

Thank You

Reference

- Robert Osserman, A Survey of Minimal Surfaces.
- Manfredodo P. Carmo, Differential Geometry of Curves and Surfaces.
- Wolfgang Kühnel, Differential Geometry Curves - Surfaces - Manifolds.