Summer Reading Program 2024 Final Report
Topic: Taylor Expansion of $(1+x)^{1/n}$

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1 Introduction

This report is written during the 2024 UBC Summer Reading Program, guided by Emanuele Bodon, who is a postdoctoral fellow at UBC.
The proof of the main result is based on [KCD], and for general facts on the p-adic numbers, we refer to [GOU].
The Taylor expansion of $(1+x)^{1/n}$ is given by

$$(1+x)^{1/n} = \sum_{k=0}^{\infty} \binom{1/n}{k} x^k,$$

where for any $a \in \mathbb{R}$,

$$\binom{a}{k} = \frac{a \cdot (a-1) \cdot \dots \cdot (a-k+1)}{k!}.$$

Observe that when $n = 2$,

$$\begin{aligned}(1+x)^{1/2} &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{64}x^4 + \dots \\ &= 1 + \frac{1}{2}x - \frac{1}{2^3}x^2 + \frac{1}{2^4}x^3 - \frac{5}{2^6}x^4 + \dots\end{aligned}$$

and when $n = 3$,

$$\begin{aligned}(1+x)^{1/3} &= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \dots \\ &= 1 + \frac{1}{3}x - \frac{1}{3^2}x^2 + \frac{5}{3^4}x^3 - \frac{10}{3^5}x^4 + \dots\end{aligned}$$

and when $n = 6$,

$$\begin{aligned}(1+x)^{1/6} &= 1 + \frac{1}{6}x - \frac{5}{72}x^2 + \frac{55}{1296}x^3 - \frac{935}{31104} + \dots \\ &= 1 + \frac{1}{6}x - \frac{15}{6^3}x^2 + \frac{55}{6^4}x^3 - \frac{8415}{6^7}x^4 + \dots\end{aligned}$$

It seems it's always possible to write $\binom{1/n}{k}$ in a fraction whose denominator is a power of n.

2 Purpose

In this report, we will show that for every positive integer n and non-negative integer k, $\binom{1/n}{k}$ can always be written as a/n^b for some $a \in \mathbb{Z}$ and some $b \in \mathbb{Z}_{\geq 0}$, or equivalently, every prime factor of the denominator of the rational number $\binom{1/n}{k}$ is a prime factor of n.

3 Background

For a fixed prime number p, We will use $\mathbb{Q}_p$ for the p-adic numbers for some prime number p. This section includes some results we will use later in the proof.
Since $\mathbb{Q}_p$ is a metric space, we can talk about continuous functions from $\mathbb{Q}_p$ to some other metric space.

Definition 3.1 (Continuous function)**.** Suppose X and Y are metric spaces, $U \subseteq X$, a function $f : U \to Y$ is said to be continuous at $a \in U$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ (possibly depending on a) such that, for every $x \in U$,

$$d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon.$$

If f is continuous at every $a \in U$, then f is said to be continuous on U.

We will denote by $|\cdot|_p$ the p-adic absolute value normalized by $|p|_p = \frac{1}{p}$, and by $|\cdot|_{\mathbb{R}}$ the usual absolute value on $\mathbb{R}$.

Example 3.1. The constant function $f(x) = c$ for some fixed $c \in \mathbb{Q}_p$ is continuous on $\mathbb{Q}_p$, since at each point $a \in \mathbb{Q}_p$, for every $\varepsilon > 0$, $|f(x) - f(a)|_p = |c - c|_p = 0 < \varepsilon$ always holds.

Example 3.2. The identity function $f(x) = x$ is continuous on $\mathbb{Q}_p$, since at each point $a \in \mathbb{Q}_p$, for every $\varepsilon > 0$, choosing $\delta = \varepsilon$ gives $|x-a|_p < \delta \Rightarrow |x-a|_p < \varepsilon \Rightarrow |f(x) - f(a)|_p < \varepsilon$.

Example 3.3. Sum of continuous functions on $\mathbb{Q}_p$ is continuous. Take two continuous functions $f, g : U \to \mathbb{Q}_p$ where U is a subset of $\mathbb{Q}_p$. By continuity of f, g, at each point $a \in U$, for all $\varepsilon > 0$, there exist $\delta_f, \delta_g > 0$ such that

$$|x - a|_p < \delta_f \Rightarrow |f(x) - f(a)|_p < \varepsilon,$$

$$|x - a|_p < \delta_g \Rightarrow |g(x) - f(a)|_p < \varepsilon.$$

Pick $\delta = \min\{\delta_f, \delta_g\}$ and by strong triangle inequality, we get

$$\begin{aligned} |x - a|_p < \delta \Rightarrow & |(f(x) + g(x)) - (f(a) + g(a))|_p \\ & \leq \max\{|f(x) - f(a)|_p, |g(x) - g(a)|_p\} < \varepsilon. \end{aligned}$$

Therefore the sum of continuous functions on $\mathbb{Q}_p$ is continuous.

Example 3.4. Product of continuous functions on $\mathbb{Q}_p$ is continuous. Take two continuous functions $f, g : U \to \mathbb{Q}_p$ where U is a subset of $\mathbb{Q}_p$. By continuity of f, g, at each point $a \in U$, for all $\varepsilon > 0$, we have

$$|x - a|_p < \delta_f \Rightarrow |f(x) - f(a)|_p < \frac{\varepsilon}{\varepsilon + |g(a)|_p}.$$

and

$$\begin{aligned}|x-a|_p<\delta' &\Rightarrow |g(x)-g(a)|_p<\varepsilon\\ &\Rightarrow |g(x)|_p \le |g(x)-g(a)|_p+|g(a)|_p<\varepsilon+|g(a)|_p.\end{aligned}$$

If $f(a)=0$, then

$$\begin{aligned}|x-a|_p<\min\{\delta',\delta_f\} \Rightarrow\ & |f(x)g(x)-f(a)g(a)|_p\\ &\le \max\{|f(x)g(x)-f(a)g(x)|_p, |f(a)g(x)-f(a)g(a)|_p\}\\ &= |g(x)|_p\cdot|f(x)-f(a)|_p\\ &\le (\varepsilon+|g(a)|_p)\frac{\varepsilon}{\varepsilon+|g(a)|_p}\\ &=\varepsilon.\end{aligned}$$

Now if $f(a)\neq 0$, then there exists δ_g such that

$$|x-a|_p<\delta_g \Rightarrow |g(x)-g(a)|_p<\frac{\varepsilon}{|f(a)|_p}.$$

Therefore, if $|x-a|_p<\min\{\delta',\delta_f,\delta_g\}$,

$$\begin{aligned}|f(x)g(x)-f(a)g(a)|_p &\le \max\{|f(x)g(x)-f(a)g(x)|_p, |f(a)g(x)-f(a)g(a)|_p\}\\ &= \max\{|g(x)|_p\cdot|f(x)-f(a)|_p, |f(a)|_p\cdot|g(x)-g(a)|_p\}\\ &< \max\{(\varepsilon+|g(a)|_p)\frac{\varepsilon}{\varepsilon+|g(a)|_p}, |f(a)|_p\frac{\varepsilon}{|f(a)|_p}\}\\ &=\varepsilon.\end{aligned}$$

Therefore the product of continuous functions on $\mathbb{Q}_p$ is continuous.

Example 3.5 (Continuity of polynomial functions)**.** Combining the previous four examples, we can conclude that all polynomial function on $\mathbb{Q}_p$ is continuous, and in particular, the functions $x\mapsto \binom{x}{k}$ for a non-negative integer k is continuous in $\mathbb{Q}_p$.

Example 3.6 (Continuity of absolute value)**.** The absolute value function

$$|\cdot|:\mathbb{Q}_p\to\mathbb{R}$$

is continuous. Indeed, for every at every point $a\in\mathbb{Q}_p$, $\varepsilon>0$, choosing $\delta=\varepsilon$ gives

$$|x-a|_p<\delta=\varepsilon \Rightarrow \big||x|_p-|a|_p\big|_{\mathbb{R}}<\varepsilon$$

by the reversed triangle inequality.
Therefore the absolute value functions is continuous.

4 Proof

Fix a positive integer n; for each non-negative integer k, let the rational number $\binom{1/n}{k}$ be written in reduced form $\frac{a}{b}$. Let p be a prime number such that $p \nmid n$; we want to show that $p \nmid b$. We will later see how this is enough to prove our result.
Consider $|\frac{1}{n}|_p$ in $\mathbb{Q}_p$, we have that $|n|_p = 1$ since $p \nmid n$, therefore we have

$$\left|\frac{1}{n}\right|_p = 1.$$

That is,

$$\frac{1}{n} \in \mathbb{Z}_p.$$

Since every element in $\mathbb{Z}_p$ is a limit of an integer Cauchy sequence ($\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with respect to the p-adic absolute value), we can find a sequence of integers (x_i) such that

$$\lim_{i\to\infty} x_i = \frac{1}{n}.$$

Now since the function $x \mapsto \binom{x}{k}$ is a polynomial which is continuous, we have

$$\binom{1/n}{k} = \binom{\lim_{i\to\infty} x_i}{k} = \lim_{i\to\infty} \binom{x_i}{k}.$$

Since each x_i is an integer, $\binom{x_i}{k}$ is an integer, and $|\binom{x_i}{k}|_p \leq 1$. By continuity of the absolute value, we get

$$\left|\frac{a}{b}\right|_p = \left|\binom{1/n}{k}\right|_p = \left|\lim_{i\to\infty} \binom{x_i}{k}\right|_p = \lim_{i\to\infty} \left|\binom{x_i}{k}\right|_p \leq 1.$$

If $p \mid b$, then since $\frac{a}{b}$ is in reduced form, we must have $p \nmid a$, and therefore $|\frac{a}{b}|_p > 1$, a contradiction. This shows $p \nmid b$.
Now let $p_1, p_2, \ldots, p_l$ be the prime divisors of n, the above argument shows that $\binom{1/n}{k}$ can be written as

$$\binom{1/n}{k} = \frac{a}{p_1^{e_1} p_2^{e_2} \ldots p_l^{e_l}},$$

where $a \in \mathbb{Z}$, $e_1, \ldots, e_l \in \mathbb{Z}_{\geq 0}$.
Suitably multiplying some $p_1, p_2, \ldots, p_l$ on the numerator and denominator, we get

$$\binom{1/n}{k} = \frac{a'}{n^e},$$

for some $a' \in \mathbb{Z}$ and $e \in \mathbb{Z}_{\geq 0}$. This completes the proof.

References

[GOU] Fernando Q. Gouvêa. *p-adic Numbers. An introduction.* Universitext. Springer, third edition, 2020.

[KCD] KCd. Show that $1 + t$ lies in $\mathbb{Z}[1/2][\![t]\!]$. Mathematics Stack Exchange, 2012. Available at `https://math.stackexchange.com/a/136288` (version: 2012-10-12).